Paradoxes and Infinities
CTY Course Syllabus

Week 1, Day 1

Morning Class

Ice-breaker (name game activity)

- Students sit in a circle, each student has to say his/her name together with all the names of the previous students.

Review honor code

Discussion:

- What do you think you will learn in this class? Why did you choose it?
- What is a paradox? Do paradoxes exist in the physical world? In math? How do paradoxes come about? Do you know of any paradoxes?

Paradox 1

- Omnipotence paradox.

Paradox 2

- The crocodile dilemma

Definition of paradox

- An argument based on (apparently) valid premises, following (apparently) valid reasoning, which leads to (apparently) contradictory or invalid conclusion
• Exploring the definition of paradox

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<th>Case</th>
<th>Assumptions</th>
<th>Reasoning</th>
<th>Conclusion</th>
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• Investigate the above table with all \(2^3\) possible cases:

1. Starting from something valid and following valid reasoning we derive something valid
2. Paradox
3. \(\frac{16}{64} = \frac{1}{4}\) (because the 6s in numerator and denominator cancel out)
4. Proof that 1=2:
   \[
   a = b \implies a^2 = ab \implies 2a^2 = a^2 + ab \implies 2a^2 - 2ab = a^2 - ab \implies 2(a^2 - ab) = a^2 - ab \implies 2 = 1
   \]
5. We are in Texas. Texas is in the US. Thus we are in the US.
6. Russell’s famous “proof” that if \(2 + 2 = 5\) then he is the pope.
7. Ask students to create their own example for this case.
8. The sky is green. When the sky is green it snows. Thus it snows.

• Discuss number of different options if we have \(n\) categories and \(m\) values for each of them \((n^m)\).
**Introduction to propositional logic**

- Define statements (sentences that are true or false). Same statements can be true or false in different worlds. For example, $11 + 2 = 1$ is a false statement in regular (arithmetic but it is true in mod12 arithmetic (time)

- Conjunction, disjunction, negation

- English to Math and vice versa

- Truth tables

- Venn diagrams

**Afternoon Class**

**Brain teaser:**

- Handcuffs puzzle ([http://britton.disted.camosun.bc.ca/jbhandcuff.htm](http://britton.disted.camosun.bc.ca/jbhandcuff.htm))

**Continue with propositional logic**

- Activity - Venn diagrams: Make two circles with a rope which will correspond to two different statements p, q. Have the students locate and step in the correct area of the Venn diagram for different statements built up from p and q.

**Evening Class**

**Modulo arithmetic**

- Transition from mod6 arithmetic to decimal and vice versa.

**Worksheet**

- Propositional logic

**Week 1, Day 2**

**Mornign Class**

**Continue with propositional logic**

**Conditionals: If P then Q (P → Q)**

- Use truth table to illustrate that “from false assumptions anything can follow”

- Inverse, Converse, Contrapositive
**Brain teaser:**

- Tillings with dominoes and straight trominos (group work):
  - Fill-in an 8x8 chessboard that is missing the two opposite corners with dominoes (can’t be done because there are only 30 black square on the board whereas there are 32 white ones and each domino occupies one black and one white square).
  - Fill-in an 8x8 chessboard that is missing one corner with trominos (can’t be done, 3-color the board with rgb colors interchangeably, so that there are 22 red, 21 green and 20 blue squares. Since each straight tromino occupies one square of each color, that means that tilling can’t be completed)

**Afternoon Class**

**Proof strategies:**

- Direct proof
- Proof by example (for statements with existential quantifiers)
  - Prove that there exists an even prime number
  - Counterexample
- Proof by contradiction:
  - Prime numbers are infinite
  - Prove that sqrt(2) is irrational
    - Lemma: Prove that if a² is even then a is even
- Proof by induction

**Activity**

- Muddy children (for details see here: http://sierra.nmsu.edu/morandi/coursematerials/MuddyChildren.html)

**Evening Class**

- Modulo arithmetic
- Worksheet: Logic
**Week 1, Day 3**

**MORNING CLASS**

**Proof by induction (for statements with universal quantifiers)**
- Inductive tiling with L-shaped trominoes: Fill-in an $8 \times 8$ chessboard missing one square (anywhere on the board) using L-shaped trominos (can be performed recursively). See [http://www3.amherst.edu/~nstarr/trom/puzzle-8by8/](http://www3.amherst.edu/~nstarr/trom/puzzle-8by8/)
  - Base case: For a $2 \times 2$ board missing one square it’s trivial
  - If you can fill-in a $2n \times 2n$ board, then for the $2n + 1 \times 2n + 1$ divide the board into 4 sub-boards and fill-in the one with the missing square inductively. For the remaining three, observe that if you remove the corner in the middle from each, you create 3 subproblems where you can fill the board inductively. Remove them by placing a tromino.

**Activity (Induction):**

This is a two player game that utilizes induction. Each of the two players is given a card with a natural number on it and place it on their foreheads. The players don’t know their own numbers but they can see the other player’s number. The two numbers on the cards are consecutive (this information is announced to the players in advance). The objective is to find the number on your forehead. The key is that the two numbers will always be consecutive.

The game is played in rounds. In each round both players are asked whether they know the number on their head. If they do they are supposed to announce their number, otherwise they have to say that they don’t.

Suppose that one of the players has card number 1. The other player can only have card number 2, so he/she can know and announce that from round 1. If this is not the case then both players will say that they don’t know their numbers and the game will continue with round 2.

During round 2, both players know that none has card number 1, so the smallest available card is now card with number 2. If one of the players has this card then the other would know that he should have number 3 (because he can’t have card with number 1). In that case he announces that he has number 3. Otherwise the game continues with round 3.

The students should prove by induction that the first player to learn their number is the one with the larger number and that, if his/her card is number $k$, this will happen after $k-1$ rounds.

**Group work:**
- Math gang robs a bank. They want to share the money.
  - The rules are the following:
    - Robbers have predetermined hierarchy (Leader is 1st, 2nd, 3rd, etc.).
    - Leader designs a plan to share the money. If it passes then they share the money, otherwise the leader is eliminated.
- Assumptions:
  
  ○ Tie favors the leader (/ plan needs a majority in order to pass)
  
  ○ If a player is indifferent then he votes ‘yea’ (/ ‘nay’)
  
- If the leader is eliminated then 2\textsuperscript{nd} person becomes leader, the 3\textsuperscript{rd} becomes 2\textsuperscript{nd} etc...

- There are 4 cases (2 cases for each assumption). Ask the students to prove inductively what is going to happen in each case.

*Unexpected test paradox*

*Polya's paradox*

**AFTERNOON CLASS**

*Inductive Proofs*

- For all \(n \geq 1\), \(1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}\)

- For all \(n \geq 1\), \(1 + 3 + 5 + \cdots + 2n + 1 = n^2\)

**EVENING CLASS**

*Worksheet: Logic*

For those who finished their worksheet we gave induction questions to work on (up to how many areas can we divide the plane with \(n\) intersecting lines, see [http://www.jimloy.com/geometry/plane.htm](http://www.jimloy.com/geometry/plane.htm))

**Week 1, Day 4**

**MORNING CLASS**

*Project*

Create a poster on your favorite paradox (research on the internet).

- Achille’s and the tortoise

- Grandfather’s paradox

- Court paradox

- Arrow paradox

- Twin Paradox
Afternoon Class

Paradoxes of induction:

  - A man with no hair is bald
  - If a man with \( n \) hairs is bald then a man with \( n+1 \) hairs is bald
  - (Introduction to fuzzy logic)
- Polya’s Paradox: [http://en.wikipedia.org/wiki/All_horses_are_the_same_color](http://en.wikipedia.org/wiki/All_horses_are_the_same_color)
  - One horse is same color as itself
  - Assume true for \( n \) horses. Also true for \( n+1 \) since first \( n \) are the same and last \( n \) are the same.

Evening Class

The students work on their posters for their paradoxes and ready themselves to present them the following day.

Week 1, Day 5

Morning Class

Continue with paradoxes of induction:
- Unexpected test paradox

Oxymorons

Two-word sentences where the first contradicts the second (like “open secret”)
- Discussion: Suggest your own example of an oxymoron

Introduction to self-referential paradoxes:
- Liar’s paradox
  - Epimenides
  - ‘This sentence is false’
- L1: L2 is false.
  L2: L1 is true.

Afternoon Class

Poster session.
Week 1, Day 6

**EVENING CLASS**
Review for Tuesday's quiz.

Week 2, Day 1

**MORNING CLASS**

*Introduction to sets*
- What is a set?
- Membership
- Sets of sets
- Listing vs. description of sets
- Finite sets vs. infinite sets (ellipse symbol ‘…’)

*Self-referential paradoxes revisited*
- Barber’s paradox
- Reference book paradox
- Grelling-Nelson paradox (Is “heterological” heterological?)
- Russell’s paradox

*Discussion*
What is common behind all the self-referential paradoxes (they all describe situations where something refers to things of its own category, so in that sense it is able to talk for itself. All have a common pattern, ask students to try and figure it out)

*Individual work*
Create your own self-referential paradox.

*Brain Teaser*
Father who left 17 camels to his boys (1/2 to the first, ¼ to the second and 1/9 to the 3rd). In order to divide the camels they had to borrow one. Then first son gets 9 camels, second gets 6 and third gets 2, which adds up to 17. Where did the last camel go?

*Preparation for Theseus’ ship debate*
Explain the paradox, let students pick their opinion.
**Afternoon Class**

*Debate (joint with philosophy class): Theseus’ Ship*

The paradox questions an object’s identity if all parts of the object are physically replaced by identical but new parts. Suppose that we replace the planks of Theseus’ ship one-by-one with new ones and we put the old ones together and build a new ship. The question now is: which is the original Theseus’ ship?

- The initial ship is the original
- The new ship is the original
- None of the ships is original
- Both ships are original

Divide students into 3 groups according to their opinion. Let students prepare for 30’ (read the book and do group discussion). Then have a debate (each of them has to talk for at most 2’, allow 1’ for questions).

**Evening Class**

- Prepare for quiz. Work on set theory problems

Week 2, Day 2

**Morning Class**

*Activity*

- *Listing vs. Description of set*. Divide students in ordered pairs. Give the first student in the pair the description of a set and ask them to come up with the listing. Hide the description and tell them to give their partner the listing and ask their partner to figure out the description of the set. When done, let them make sure that the description given to the first student agrees with the one the second student came up. If they don’t try to figure out what went wrong in the process. Prolong this activity as long as it feels right (faster students can try to swap duties or try to repeat the process with more difficult patterns).

- Activity: Find your match. Students were given cards with sets (listings or descriptions) and they were supposed to find all the other people in the room that match with (have sets that are equal with them). There might be more than 2 people that match. After finding a group ask the students to give yet another description of the set that they represent.

- Individual assignment (for faster students): Write Russell’s paradox in set-theoretic notation.

*Sets continued*

- Listing vs. description of a set
- Venn Diagrams
• Subsets of sets - Powerset

• Sets of natural numbers $\mathbb{N}$, integer numbers $\mathbb{Z}$, rational numbers $\mathbb{Q}$ and real numbers $\mathbb{R}$

• When are two finite sets equinumerous? One-to-one and onto correspondence

**Discussion:**

Is the definition of equinumerous sets reasonable?

• When are two infinite sets equinumerous? (show that Odd numbers $\sim$ Even numbers). Negative integers equinumerous with positive integers.

• More equinumerousness:
  
  o $N \sim N \cup \{0\}$
  
  o $N \sim 2N$
  
  o $N \sim N \times N$

• Prove that $N \sim 2N$

• **Individual work:**
  
  o $N \sim \mathbb{Z}$ (explanation in class)
  
  o $N \sim \mathbb{Q}$ (explanation in class)

**AFTERNOON CLASS**

**Hilbert Hotel:**

• $N \sim N \cup \{0\}$

• $N \sim 2N$ (explanation in class)

• $N \sim N \times N$ (explanation in class)

**EVENING CLASS**

• Worksheet on set theory
Week 2, Day 3

Morning Class

Review countable sets and correspondences

Intro to diagonalization

Activity:

Dodgeball 2-player game (to understand diagonalization).

- Rules: Player1 (the baller) has a $6 \times 6$ array to fill-in and player2 (the dodger) has an $1 \times 6$ row. Each player takes turns as described below:
  - Baller begins by filling-in the first row of his array with Xs and Os, one in each square.
  - Then the dodger puts an X or an O in the first box of their row.
  - Repeat the above steps until all rows in baller’s array and all boxes in dodger’s row are competed.

- Winner: The baller wins if any of his rows are identical to the dodger’s row. The dodger wins if his row is different from all the rows of the baller.

- Questions (explain what is a strategy):
  - Figure out that 2nd player has a strategy to win.
  - Figure out that if 2nd player makes a mistake in the 1st round, then 1st player can always win.

Give definition of countable sets.

Afternoon Class

Diagonalization (continued)

- Interval $(0,1)$ is not equinumerous to $\mathbb{N}$ – Diagonalization
- Define uncountable set, give intuition (not listable)
- Interval $(a,b)$ is equinumerous to $(0,1)$
- Set of real numbers is equinumerous to $(0,1)$

Evening Class

- Worksheet on Set theory
Week 2, Day 4

Morning Class

Uncountable sets revisited
- Set of predicates ($N \to \{0,1\}$ functions) is uncountable. Proof by diagonalization
- $P(N)$ (the powerset of natural numbers) is uncountable. $1$ to $1$ correspondence with previous set (explain characteristic function)
- Practice on diagonalization

Infinitely many levels of infinity
- Discuss concept
- For really fast students: Prove that that powerset of an infinite set is a larger infinity

Discussion: Why are these discoveries important (consequences in math and CS)
- Set of programs is countable
- Set of problems (predicates) is uncountable.
- Thus there are problems that computers can’t solve

Afternoon Class

Monty Hall Paradox
- Describe the situation. Ask about students’ intuition.
- Activity: Divide the students in pairs. Each pair is given 3 cards (one stands for the car, the others for the goats.) The students switch off playing host or guest and record their results (repeat the experiment for 30 times if player sticks to their original choice and 30 if they decided to switch boxes). Sum up the results of all the experiments. It should be clear that switching is a much better strategy.
- Explain the reason why this happens.
- Repeat experiment with 52 cards. Player picks a card, host opens remaining 50 and asks player if he wants to switch with the last unopened card. It should be clear that the original card has probability only $\frac{1}{52}$ to contain the car, so it’s better to switch.

Evening Class
- Quiz 2
- Worksheet: set theory
Week 2, Day 5

Morning Class

Project:
Create power-point presentation for a famous mathematician related to our discussion.

- Cantor
  - Uncountability
  - Continuum Hypothesis

- Russel
  - Russell’s paradox
  - Principia mathematica and his quest to axiomatize math

- Peano

- Zermelo
  - ZF axioms
  - Axiom of Choice
  - Banach Tarski Paradox

- Hilbert’s talk in Paris in 1900
  - Continuum Hypothesis,
  - Axiom of Choice,
  - 10th problem
  - Ignorabimus

- Kurt Gödel
  - Metamathematics
  - Incompleteness theorems and their importance
  - Continuum Hypothesis
● Alan Turing
  ○ Turing Machine
  ○ Turing Test
  ○ Enigma
  ○ Halting Problem – Undecidability

**AFTERNOON CLASS**

Project continued: Poster presentation (fill-in with explanations and details when needed)

**Week 2, Day 6**

**EVENING CLASS**
  ● Finish worksheets on set theory

**Week 3, Day 1**

**MORNING CLASS**

*Introduction to Probabilities*
  ● Definitions:
    ○ Experiment- situation that involves chance
    ○ Outcome- the result of a single trial
    ○ Event – One or more outcomes of an experiment
    ○ Probability- Measures how likely an experiment is

  ● Example:
    ○ Experiment: Roll a die.
    ○ Possible outcomes: 1,2,3,4,5,6
    ○ Event: Die lands on 5
    ○ Probability: 1/6
• Probability of an event $A$:
\[
Pr(A) = \frac{\text{# of good outcomes}}{\text{# of all outcomes}}
\]

• Example 2: Jar with marbles: 7 white marbles, 5 purple marbles. What is the probability of getting a white marble?

• Example 3: Roll 2 dice
  ○ Draw chart on blackboard
  ○ Ask multiple questions regarding the probability of different events

**Activity:**
Flip a coin 30 times and report results. Then produce 30 fake coin flips. Compare them and figure out possible ways to discover the fake coin flips.

**Afternoon Class**

**Conditional probability**
• Definition of conditional probability. Bayes’ theorem
  ○ Back to the two-dice chart, explain conditional probabilities using set notation.
  ○ Independent work: Ask students to compute conditional probabilities.

**EVENING CLASS**
• **Worksheet**: Probabilities

**Week 3, Day 2**

**MORNING CLASS**

**Venn Diagrams**
Return to the chart of probability on 2 rolled dice.
• Probabilities and set theory
  ○ Explore the probabilities of the union and intersection of different events (De Morgan’s Law).
  ○ Independent work: Ask students to compute probabilities of similar events False Positive paradox

• Expectations
AFTERNOON CLASS
- More expectations - Two envelopes paradox

EVENING CLASS
- Worksheet: Probabilities - Expectations

Week 3, Day 3

MORNING CLASS

Zeno’s Paradox:
- Resolution 1: Space is quantized (there is a smallest possible distance (Plank distance) in the universe that can’t be halved.
- Resolution 2: The sub-distances to be covered form an infinite series which adds up to the finite distance. (Discuss concept of limit, talk about series)

Infinite series
- Convergence
- Self-similar infinite series
- Fractals

AFTERNOON CLASS

Activity (together with game theory): Casino day
- Designed 6 games of chance with expectations and probabilities to win easy to compute. Let students play the games.
- Computed the probabilities and expectations for the casino games.

EVENING CLASS

Worksheets:
- Infinite Series - fractals
Week 3, Day 4

Morning Class

Instructor-TA evaluation

Post-assessment

Continue with probabilistic Paradoxes:
- Raven’s Paradox
- St. Petersburg Paradox

Afternoon Class

Is universe finite or infinite?:
- Olber’s paradox: [http://en.wikipedia.org/wiki/Olbers’_paradox](http://en.wikipedia.org/wiki/Olbers’_paradox), [http://www.youtube.com/watch?v=yQz0VgMNGPQ](http://www.youtube.com/watch?v=yQz0VgMNGPQ)
- Talk by Jeff Weeks “The shape of space” (from the ‘Math Encounters’ series)
- During the presentation, explain concepts as necessary:
  - What is the difference of infinite and unbounded
  - What is a torus
  - Play games on torus topology, using Jeff Weeks’ online application
  - Explain what a 3d torus is and what it means for a shape to be curved in another dimension.
- Continue with article
  - Redo the problem as suggested in the comments of the article
  - Introduce false positive paradox
  - Compute probabilities using the suggested approach
  - Give intuition (the false positives dominate)
Week 3, Day 5

Morning Class

Dimensions:
- Watch “Flatland” movie
- Explain the geometry described in the movie
- Introduce 4th dimension:
  - Time as 4th dimension
  - Spatial 4th dimension:
    - Can we imagine the 4th dimension? Analogy with Flatland movie
    - Tesseracts:
      - Explain what a tesseract is
      - Drawing a tesseract in 2d
      - Projection of tesseract in 3d
      - Unfolding of a tesseract (try to imagine how it folds)

More dimensions (hypercubes):

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